Thermally driven motion of highly viscous fluids

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Abstract—Using a formal, unified method described in a previous paper, several sets of equations are derived governing the thermally driven motion of highly viscous fluids. The cases covered are (i) a strongly heated fluid; (ii) a strongly heated, shallow fluid; (iii) a strongly heated, deep liquid with a small coefficient of volume expansion; (iv) a weakly heated, deep fluid; (v) a weakly heated, deep fluid with large Prandtl number; and (vi) a strongly heated, deep liquid with large Prandtl number and small coefficient of volume expansion. These cases are distinguished by the orders of magnitude of the following five parameters: the Prandtl number, the nondimensionalized depth of the fluid, the nondimensionalized coefficient of volume expansion, the 'absolute' Grashof number, and the heat addition parameter. In all six sets of equations obtained, the inertia terms are absent from the momentum equations. The thermal inertia terms are absent from the energy equations for cases (i), (ii) and (iii). The unperturbed situations arising in cases (iii)–(vi) are supposed to be steady and quiescent. Corresponding unperturbed solutions are presented in detail.

1. INTRODUCTION

IN A RECENT paper [1], a formal, unified method was used to derive sets of approximate equations governing the thermally driven motion of a strongly heated, shallow fluid, and of a strongly heated, deep liquid with a small coefficient of volume expansion. Additional limits considered were a weakly heated, deep fluid and a weakly heated, shallow fluid. The scaling used in all these cases was based on the assumption that inertial effects are of importance in at least part of the flow. It was noted that a different scaling should be used for highly viscous flows, such as occur in the thermally driven motion of geophysical fluids, or of other highly viscous fluids. Such other scalings have been used by McKenzie *et al.* [2], Jarvis and McKenzie [3], and Turcotte *et al.* [4]. McKenzie *et al.* [2] solved the resulting equations numerically, in order to investigate convection in the earth's mantle. Jarvis and McKenzie [3] studied convection in a deep, compressible fluid. Turcotte *et al.* [4] presented multiple solutions for flows in an internally heated, vertical channel with viscous dissipation and pressure work. A number of other authors who were interested in the effects of viscous dissipation, variations on viscosity, pressure work, etc. [5–12] used scalings similar to that of ref. [1]. Ostrach [5] considered internal viscous flows with body forces, and also presented a comprehensive review of laminar flows with body forces [6]. Torrance and Turcotte [7] studied the influence on thermal convection of large variations in viscosity. Turcotte *et al.* [8] investigated the influence of viscous dissipation on Bénard convection, for a fluid with zero compressibility. Parmentier and Torrance [9] carried out numerical calculations concerning the influence of a rigid boundary on mantle convection. Oxburgh and Turcotte [10] discussed the relation between viscous flows and mechanisms of continental drift. Peltier [11] developed a linear stability theory for deep fluids, with applications to the planetary mantle. Velarde and Cordon [12] considered the basic equations applicable to a weakly heated, deep fluid, and discussed the role of viscous dissipation. Further references to related papers may be found in the works cited. The relations between several of the works cited and the present paper are discussed briefly in the Appendix.

The purpose of the present work is to extend the formal method presented in ref. [1] to flows that are highly viscous. The fundamental equations of fluid dynamics are scaled using characteristic velocities appropriate for such flows. The Prandtl number is taken to be either of order 1 (Sections 3 and 4), or very large (Sections 5 and 6). Section 3 concerns strongly heated fluids, which can be further specified as being shallow (Section 3.1) or as having a small coefficient of volume expansion (Section 3.2). Sections 4 and 5 treat weakly heated, deep fluids. Finally, Section 6 contains results for a strongly heated, deep liquid with a small coefficient of volume expansion.

2. BASIC EQUATIONS

The starting point of the present work is the same as used in ref. [1], and consists of the fundamental equations of fluid dynamics (see, e.g. [13, Table 10.4-1]) written in the form

$$\partial \rho / \partial t + \nabla \cdot (\rho \mathbf{V}) = 0 \text{ (continuity)} \quad (1)$$

$$\rho \left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} \right]$$

$$= -\frac{1}{\delta \varepsilon} \nabla P - \mathbf{k} \frac{\rho}{\delta} - \frac{v_m}{\beta L} [\nabla \cdot \tau] \quad \text{(momentum)} \quad (2)$$

NOMENCLATURE

c_m $(p_m/\rho_m)^{1/2}$

c_v specific heat at constant volume, nondimensionalized by c_{vm}

c_p specific heat at constant pressure, nondimensionalized by c_{pm}

g acceleration of gravity

Gr Grashof number, see equation (4a) or (4b)

$\mathbf{k}$ unit vector in z-direction

K isothermal compressibility, $\rho^{-1}(\partial\rho/\partial p)_\theta$

L length scale

p_m pressure

P pressure, nondimensionalized by p_m

Pr Prandtl number, $\mu c_p/\kappa$

q_m heat addition parameter, $Q_m L^2/(\kappa_m\theta_m)$

Q_m characteristic rate of heat addition per unit volume

Q rate of heat addition per unit volume, nondimensionalized by Q_m

r_m absolute Grashof number, $\alpha_m\theta_m gL^3/v_m^2$

s entropy, nondimensionalized by $s_m \equiv c_{pm}\theta_m$

t time, nondimensionalized by L/β

T temperature, nondimensionalized by θ_m

$\mathbf{V}$ velocity, nondimensionalized by β

z vertical coordinate, nondimensionalized by L.

Greek symbols

α coefficient of volume expansion, $-\rho^{-1}(\partial\rho/\partial\theta)_p$

β characteristic velocity

γ ratio of specific heats

δ $\beta^2/(gL)$

ε nondimensionalized depth, gL/c_m^2

θ temperature

κ thermal conductivity, nondimensionalized by κ_m

μ dynamic viscosity, nondimensionalized by μ_m

v kinematic viscosity, nondimensionalized by v_m

ρ density, nondimensionalized by ρ_m

τ viscous part of stress tensor, nondimensionalized by $\mu_m\beta/L$

ϕ $c_m^2/(c_{pm}\theta_m)$.

Subscripts

m constant reference quantities

z component in the z-direction

0, 1, 2 order in perturbation expansion.

$$\rho c_p\left[\frac{\partial T}{\partial t}+(\mathbf{V}\cdot\nabla)T\right]=\frac{1}{Pr_m}\frac{v_m}{\beta L}\nabla\cdot(\kappa\nabla T)$$

$$+\alpha\theta\phi\left[\frac{\partial P}{\partial t}+(\mathbf{V}\cdot\nabla)P\right]+\frac{Q_m L}{\rho_m c_{pm}\theta_m\beta}Q$$

$$-\delta\varepsilon\phi\frac{v_m}{\beta L}(\tau:\nabla\mathbf{V})\quad\text{(energy)}.\quad(3)$$

The characteristic velocity β is as yet unspecified, while $\delta\equiv\beta^2/(gL)$. The parameter $\varepsilon\equiv gL/c_m^2$ represents the ratio of the physical depth L to the characteristic hydrostatic depth c_m^2/g. The parameter $\phi=c_m^2/(c_{pm}\theta_m)$ equals $(\gamma_m-1)/\gamma_m$ for an ideal gas, and is of the order of 10^{-4} for liquids at a pressure of 0.1 MPa and at room temperature. It is assumed here that conditions are such that ϕ is at most of order 1. The product $\alpha\theta$ is considered to be of order 1 in the expansion procedures of the following Sections 3–5, and to be of the same order as the small parameter $\alpha_m\Delta\theta$ in Section 6. Equations (1)–(3) must be supplemented by the thermal equation of state $\rho=\rho(P,T)$, which is assumed to be given. Also assumed given are the dependence of the specific heat and the various transport coefficients on temperature. Their dependence on pressure is neglected, although this dependence could readily be taken into account if needed. The motion is supposed to be driven by either a heat source distribution ($Q_m\neq0$, $\Delta\theta=0$), or by a prescribed temperature difference at the boundaries ($Q_m=0$, $\Delta\theta\neq0$). In the following, these two cases are referred to as case (a) and case (b), respectively.

In ref. [1], the parameter δ for case (a) was set equal to $\alpha_m\theta_m$ times the nondimensionalized strength of the heat source. This yielded the characteristic velocity, β. The value of the quantity $v_m/(\beta L)$ was supposed to be at most of order 1. For situations of present interest, the latter quantity is supposed to be large compared with 1. In developing sets of equations describing the motion of the fluid, all quantities are expanded in terms of a small parameter. Various choices are possible for this small parameter, and for β; once these have been chosen, all other results follow in a systematic way. The conditions of validity for the results obtained are specified in terms of the following dimensionless parameters: the Prandtl number Pr_m, the nondimensionalized depth ε, the nondimensionalized coefficient of volume expansion $\alpha_m\theta_m$, the absolute Grashof number $r_m\equiv\alpha_m\theta_m gL^3/v_m^2$ and the heat addition parameter $q_m\equiv Q_m L^2/(\kappa_m\theta_m)$. The parameter r_m is called the 'absolute' Grashof number because it is based on the absolute temperature θ_m. In the following pages all of these parameters are taken to be of order 1, except when indicated otherwise. Table 1 lists their order of magnitude for the six cases discussed in the following sections. Following ref. [1], the Grashof number for case (a) is defined as

$$Gr_m\equiv\frac{L^2}{v_m^2}\left(\frac{\alpha_m gL^2 Q_m}{\rho_m c_{pm}}\right)^{2/3}=(q_m r_m/Pr_m)^{2/3}\quad(4a)$$

while for case (b)

$$Gr_m\equiv\alpha_m gL^3\Delta\theta/v_m^2=r_m\Delta\theta/\theta_m.\quad(4b)$$

Table 1. Summary of orders of magnitude of parameters

	Pr_m	ε	$\alpha_m\theta_m$	r_m	q_m	$\beta L/v_m =$	$\delta =$
Strongly heated fluid (Section 3, main part)	$O(1)$	$O(1)$	$O(1)$	$\ll 1$	$O(1)$	r_m	$\alpha_m\theta_m r_m$
Strongly heated, shallow fluid (Section 3.1)	$O(1)$	$\ll 1$	$O(1)$	$\ll 1$	$O(1)$	r_m	$\alpha_m\theta_m r_m$
Strongly heated, deep liquid (Section 3.2).	$O(1)$	$O(1)$	$\ll 1$	$\ll 1$	$O(1)$	r_m	$\alpha_m\theta_m r_m$
Weakly heated, deep fluid (Section 4)	$O(1)$	$O(1)$	$O(1)$	$O(1)$	$\ll 1$	q_m/Pr_m	$\alpha_m\theta_m q_m^2/(Pr_m^2 r_m)$
Weakly heated, deep fluid (Section 5)	$\gg 1$	$O(1)$	$O(1)$	$\gg 1/Pr_m$	$\ll 1$	$(q_m r_m/Pr_m^2)^{1/3}$	$\alpha_m\theta_m(q_m^2/Pr^4 r_m)^{1/3}$
Strongly heated, deep liquid (Section 6)	$\gg 1$	$O(1)$	$\ll 1$	$O(Pr_m^{-1})$ $(\ll 1)$	$O(1)$	r_m	$\alpha_m\theta_m r_m$

The heating parameter q_m may be replaced by the Grashof number, using the identity $q_m = Pr_m Gr_m^{3/2}/r_m$; expressions with Gr_m are valid for both case (a) and case (b)—see equations (4a) and (4b).

The latter expression for Gr_m is obtained from the former if q_m is replaced by $Pr_m r_m^{1/2}(\Delta\theta/\theta_m)^{3/2}$. This replacement transforms all following results for case (a) into results for case (b), with the understanding that the heat source term Q in the energy equation must be set equal to zero in the latter case. For all cases considered, $Gr_m \ll 1$ and $\beta/a \ll 1$, where a is the speed of sound. As a result of the latter inequality, none of the sets of equations derived admits acoustic waves. This greatly reduces the difficulty of solving the equations numerically, as compared with solving the full equations of motion for compressible fluids (cf. [1] and references cited therein).

For future use, it is noted that expansion of quantities $g(T)$ in powers of a small parameter yields to the lowest two orders $g_0(T_0) = g(T_0)$, $g_1(T_0, T_1) = T_1\, dg(T_0)/dT_0$. Furthermore, it is noted that the unperturbed states arising in sections 3.2–6 are assumed to be steady and quiescent.

3. STRONGLY HEATED FLUID

Equations of motion governing the thermally driven flow of strongly heated, highly viscous fluids can be obtained by setting

$$\beta = \alpha_m\theta_m g L^2/v_m(= r_m v_m/L) \tag{5}$$

from which it follows that $\delta = (\alpha_m\theta_m)^2 g L^3/v_m^2$ $(=\alpha_m\theta_m r_m)$. Expanding in powers of δ under the assumption $r_m \ll 1$ yields to lowest order

$$\partial\rho_0/\partial t + \nabla\cdot(\rho_0 \mathbf{V}_0) = 0 \tag{6}$$

$$0 = -\frac{1}{\varepsilon\alpha_m\theta_m}\nabla P_0 - \mathbf{k}\frac{\rho_0}{\alpha_m\theta_m} - [\nabla\cdot\tau_0] \tag{7}$$

$$0 = \nabla\cdot(\kappa_0\nabla T_0) + q_m Q. \tag{8}$$

The pressure term appearing in equation (7) contains P_0, which is related to ρ_0 and T_0 by $\rho_0 = \rho(P_0, T_0)$. This pressure term can be decoupled from T_0 and ρ_0 by using a second expansion, with either ε or $\alpha_m\theta_m$ as the expansion parameter.

3.1. Strongly heated, shallow fluid
If the fluid is shallow, all quantities can be expanded in terms of ε:

$$f_0(\mathbf{r}, t; \varepsilon) = f_{00}(\mathbf{r}, t) + \varepsilon f_{01}(\mathbf{r}, t) + O(\varepsilon^2).$$

This yields

$$\nabla P_{00} = 0 \tag{9}$$

$$\partial\rho_{00}/\partial t + \nabla\cdot(\rho_{00}\mathbf{V}_{00}) = 0 \tag{10}$$

$$0 = -\frac{1}{\varepsilon\alpha_m\theta_m}\nabla P_{01} - \mathbf{k}\frac{\rho_{00}}{\alpha_m\theta_m} - [\nabla\cdot\tau_{00}] \tag{11}$$

$$0 = \nabla\cdot(\kappa_{00}\nabla T_{00}) + q_m Q. \tag{12}$$

It follows from the first of these equations that the lowest order pressure is uniform: $P_{00} = P_{00}(t)$. The resulting value of P_{00} depends on the conditions under which the fluid is maintained. Equation (12) is simply the steady-state heat conduction equation. The density differences resulting from the heat addition drive the flow as specified by equations (11) and (10), i.e. without inertial effects making themselves felt. When Q is independent of time, all other quantities are independent of time also. 'Sudden' changes in Q (i.e. changes that are sudden with respect to the time scale L/β) cause immediate changes in temperature, density, pressure and velocity. The terms of the continuity equation (10) yield Dirac delta functions in response to sudden changes in Q. Gradual changes in Q cause gradual changes in the flow; the corresponding values of the terms of (10) are finite.

3.2. Strongly heated, deep liquid with $\alpha_m\theta_m \ll 1$
For a liquid with a small coefficient of volume expansion $(\alpha_m\theta_m \ll 1)$, expansion of equation (7) in powers of $\alpha_m\theta_m$ yields to lowest order

$$dP_{00}/dz = -\varepsilon\rho_{00}. \tag{13}$$

Similarly, the differential equation of state $d\ln\rho = Kp_m\, dP - \alpha\theta_m\, dT$ yields

$$d\ln\rho_{00} = K_{00}p_m\, dP_{00}. \tag{14}$$

The solution of equations (13) and (14) for the case $K = \text{constant} = K_m$ is

$$\rho_{00} = \exp[a(P_{00} - 1)] = (1 + \varepsilon az)^{-1} \quad (14a)$$

where $a = K_m p_m$ [1]. Similarly, for $K P p_m = \text{constant} = K_m p_m = a$ the solution is

$$\rho_{00} = P_{00}^a = [1 - \varepsilon(1 - a)z]^{a/(1-a)}. \quad (14b)$$

Expansion of equations (6)–(8) yields the equations governing the perturbed state:

$$\nabla \cdot (\rho_{00} \mathbf{V}_{01}) = 0 \quad (15)$$

$$0 = -\varepsilon^{-1} \nabla P_{01} - \mathbf{k}\rho_{01} - [\nabla \cdot \tau_{01}] \quad (16)$$

$$0 = \nabla \cdot (\kappa_{00} \nabla T_{00}) + q_m Q. \quad (17)$$

The lowest order velocity here is denoted by $\mathbf{V}_{01}$ rather than by $\mathbf{V}_{00}$ because it does not belong to the unperturbed state. The relation between ρ_{01}, P_{01} and T_{00} is obtained by expanding the differential equation of state, and integrating. For $K = \text{constant} = K_m$ this yields

$$\rho_{01} = \rho_{00}\left[K_m p_m P_{01} - \int_1^{T_{00}} (\alpha_{00}/\alpha_m) \, dT_{00}' \right]. \quad (18)$$

4. WEAKLY HEATED, DEEP FLUID

Setting $\beta L/v_m = q_m/Pr_m$, and expanding equations (1)–(3) in powers of this quantity under the assumption $q_m \ll 1$ yields

$$dP_0/dz = -\varepsilon\rho_0 \quad (19)$$

$$(d/dz)(\kappa_0 \, dT_0/dz) = 0 \quad (20)$$

$$\nabla \cdot (\rho_0 \mathbf{V}_1) = 0 \quad (21)$$

$$0 = -\frac{r_m}{\varepsilon\alpha_m\theta_m} \nabla P_1 - \mathbf{k}\frac{r_m\rho_1}{\alpha_m\theta_m} - [\nabla \cdot \tau_0] \quad (22)$$

$$\rho_0 V_{1z}[c_{p0}(dT_0/dz) + \varepsilon\phi(\alpha\theta)_0] = Pr_m^{-1}\nabla \cdot (\kappa_0 \nabla T_1)$$
$$+ Pr_m^{-1}(\partial/\partial z)(\kappa_1 \, dT_0/dz) + Q \quad (23)$$

$$\rho_1 = \rho_0(K_0 p_m P_1 - \alpha_0 \theta_m T_1). \quad (24)$$

The characteristic velocity, β, is now given by

$$\beta = Q_m L \, v_m/(\kappa_m\theta_m Pr_m) = q_m v_m/(L Pr_m) \quad \text{(case a)} \quad (25a)$$

or

$$\beta = (\alpha_m\theta_m gL)^{1/2}(\Delta\theta/\theta_m)^{3/2}$$
$$= r_m^{1/2}(\Delta\theta/\theta_m)^{3/2} v_m/L \quad \text{(case b)} \quad (25b)$$

while

$$\delta = \beta^2/(gL) = q_m^2 v_m^2/(Pr_m^2 gL^3)$$
$$= \alpha_m\theta_m q_m^2/(Pr_m^2 r_m) \quad \text{(case a)}$$

or

$$\delta = \alpha_m\theta_m(\Delta\theta/\theta_m)^3 \quad \text{(case b)}.$$

The unperturbed state is governed only by the barometric pressure equation (19), the condition of

constant heat flux (20) and the thermal and caloric equations of state. The unperturbed state therefore does not have to be the 'adiabatic state', as in the following section. Specifically, any temperature gradient obeying the relation $dT_0/dz = 1/\kappa_0(T_0)$ may be added to the unperturbed state.

5. WEAKLY HEATED, DEEP FLUID WITH $Pr_m \gg 1$

The expansion parameter used for this case is $\delta Pr_m/(\alpha_m\theta_m)$. For case (a) this parameter is set equal to $Q_m L/(\rho_m c_{pm}\theta_m\beta)$, which is the coefficient of Q in equation (3). For case (b) it is set equal to $Pr_m^{1/3} \Delta\theta/\theta_m$. It follows that the characteristic velocity now is given by

$$\beta = (Q_m gL^2 \alpha_m/\rho_m c_{pm} Pr_m)^{1/3}$$
$$= (q_m r_m/Pr_m^2)^{1/3} v_m/L \quad \text{(case a)} \quad (26a)$$

or

$$\beta = (\alpha_m gL\Delta\theta)^{1/2}/Pr_m^{1/3} \quad \text{(case b)} \quad (26b)$$

while

$$\delta = (Q_m\alpha_m/\rho_m c_{pm} Pr_m)^{2/3}(L/g)^{1/3}$$
$$= \alpha_m\theta_m(q_m^2/Pr_m^4 r_m)^{1/3} \quad \text{(case a)} \quad (27a)$$

or

$$\delta = \alpha_m\Delta\theta/Pr_m^{2/3} \quad \text{(case b)}. \quad (27b)$$

Equivalent expressions, applicable to both case (a) and case (b), are $\beta L/v_m = Gr_m^{1/2}/Pr_m^{1/3}$, $\delta = \alpha_m\theta_m Gr_m/(Pr_m^{2/3} r_m)$. It is supposed that the quantity $Pr_m\beta L/v_m = Pr_m^{2/3}Gr_m^{1/2}$ is of order 1. Combining this condition with the condition $\delta Pr_m/(\alpha_m\theta_m) = Pr_m^{1/3}Gr_m/r_m \ll 1$ yields $1/(Pr_m r_m) \ll 1$, $Pr_m Gr_m^{3/2}/r_m \ll 1$. The latter condition is equivalent to $q_m \ll 1$ for case (a), and to $Pr_m r_m^{1/2}(\Delta\theta/\theta_m)^{3/2} \ll 1$ for case (b). It follows furthermore that $v_m\delta/(\beta L) = O(\alpha_m\theta_m Pr_m Gr_m^{3/2}/r_m) \ll 1$; this result is used in the expansion procedure.

5.1. The unperturbed state

Expanding equations (1)–(3) yields to lowest order (cf. [1])

$$dP_0/dz = -\varepsilon\rho_0 \quad (28)$$

$$c_{p0} \, dT_0/dz = -(\alpha\theta)_0\varepsilon\phi \quad (29)$$

$$(d/dz)(\kappa_0 \, dT_0/dz) = 0. \quad (30)$$

As noted in ref. [1], equations (29) and (30) can be solved if, and only if,

$$(\kappa\alpha\theta/c_p)_0 = \text{constant}. \quad (31)$$

Assuming this to be the case, it follows from the thermodynamic identity

$$ds_0 = c_{p0} \, dT_0 + (\alpha\theta)_0\phi\rho_0^{-1} \, dP_0 \quad (32a)$$

together with equations (28) and (29) that $ds_0/dz = 0$. This means that the unperturbed state must be the so-

called adiabatic state. Using an alternative form of the thermodynamic identity (32a)

$$ds_0 = \gamma_m^{-1} c_{v0} \, dT_0 - (\alpha\theta)_0 (Kp)_0^{-1} \theta(P_0/\rho_0^2) \, d\rho_0 \quad (32b)$$

together with equations (28) and (30) leads to

$$(d/dz)\rho_0^{-1} = \varepsilon K p_m / \gamma(T). \quad (33)$$

This equation can be solved after the thermal and caloric equations of state have been specified. The results completely determine the dependence of ρ_0, P_0 and T_0 on z. In the following three subsections, solutions are presented for fluids with $\gamma = $ constant $= \gamma_m$, obeying a power law equation of state (Section 5.1.1), an exponential equation of state (Section 5.1.2), and equations of state that are partly power law, partly exponential (Section 5.1.3).

5.1.1. *Power law equation of state.* If $Kp = $ constant $= K_m p_m \equiv a$ and $\alpha\theta = $ constant $= \alpha_m \theta_m \equiv b$, the equation of state is given by

$$\rho = P^a T^{-b}. \quad (34)$$

It then follows from equation (31) that

$$c_{p0} = \kappa_0 \quad (35)$$

where use was made of the reference conditions $c_{p0} = \kappa_0 = 1$ at $T_0 = 1$. Assuming furthermore that the ratio of specific heats $\gamma(T) = $ constant $= \gamma_m$, use of equations (28) and (33) leads to

$$v_0 (d^2/dz^2) v_0 - (\gamma_m/a)(dv_0/dz)^2 = 0 \quad (36)$$

where $v_0 = 1/\rho_0$. Solving this equation under the boundary conditions $v_0 = 1$, $P_0 = 1$ and $T_0 = 1$ at $z = 0$ yields, after some algebra,

$$\rho_0 = [1 - \varepsilon(\gamma_m - a)z/\gamma_m]^{a/(\gamma_m - a)} \quad (37)$$

$$P_0 = [1 - \varepsilon(\gamma_m - a)z/\gamma_m]^{\gamma_m/(\gamma_m - a)} \quad (38)$$

$$T_0 = [1 - \varepsilon(\gamma_m - a)z/\gamma_m]^{(a/b)(\gamma_m - 1)/(\gamma_m - a)}. \quad (39)$$

It now follows from equation (30) together with the condition $\kappa_0 = 1$ at $z = 0$ that

$$\kappa_0 = T_0^{-1 + (b/a)(\gamma_m - a)/(\gamma_m - 1)}. \quad (40)$$

For an ideal gas, $a = b = 1$ and $\phi = 1 - 1/\gamma_m$. Equations (40) and (35) together with the assumption $\gamma = $ constant then yield $c_{p0} = c_{v0} = \kappa_0 = 1$. For a liquid under 'normal' conditions ($p_m = 0.1\,\mathrm{MPa}, \theta_m = 300\,\mathrm{K}$), $a = 10^{-4}$, $b = 0.03$–0.3, while $\gamma_m = 1$. It then follows that $\rho_0 \approx 1 + a \ln(1 - \varepsilon z)$, $P_0 \approx 1 - \varepsilon z$, $T_0 \approx 1 + (a/b)(\gamma_m - 1)\ln(1 - \varepsilon z)$, $\kappa_0 \approx 1 - \varepsilon z \approx T_0^{(b/a)/(\gamma_m - 1)}$. The order of magnitude of $\gamma_m - 1$ can be estimated from the thermodynamic identity

$$\gamma - 1 = \gamma(b^2/a)\phi P/(\rho T c_p) \approx \phi b^2/a.$$

It follows that $(b/a)/(\gamma_m - 1) \approx 1/(b\phi) \gg 1$, so that there is a very strong dependence of κ_0 on T_0. Unless ε is close to 1, this dependence extends only over a very limited range of T_0.

5.1.2. *Exponential equation of state.* If $\alpha = $ constant $= \alpha_m$ and $K = $ constant $= K_m$, the equation of state is given by

$$\rho = \exp[K_m p_m (P - 1) - \alpha_m \theta_m (T - 1)]. \quad (41)$$

Equation (31) then becomes

$$c_{p0} = \kappa_0 T_0. \quad (42)$$

Again assuming that $\gamma = $ constant $= \gamma_m$, it is found from equations (28), (33) and (41) that

$$\rho_0 = (1 + \varepsilon K_m p_m z/\gamma_m)^{-1} \quad (43)$$

$$P_0 = 1 - (\gamma_m/K_m p_m) \ln(1 + \varepsilon K_m p_m z/\gamma_m) \quad (44)$$

$$T_0 = 1 - (\gamma_m - 1)(\alpha_m \theta_m)^{-1} \ln(1 + \varepsilon K_m p_m z/\gamma_m). \quad (45)$$

Use of equation (30) finally leads to the following relation for $\kappa_0 = c_{p0}/T_0$:

$$\kappa_0 = \exp[-\alpha_m \theta_m (T_0 - 1)/(\gamma_m - 1)]. \quad (46)$$

Equations (43)–(45) assume simplified forms for a liquid under 'normal' conditions ($K_m p_m \ll 1$). For such a liquid, $\alpha_m \theta_m/(\gamma_m - 1)$ is of order $1/(\alpha_m \theta_m)$. Equation (46) thus indicates a strong dependence of κ_0 on T_0, with κ_0 decreasing as T_0 is increasing.

5.1.3. *Equations of state that are part exponential, part power law.* If $\alpha = $ constant $= \alpha_m$ and $Kp = $ constant $= K_m p_m \equiv a$, the equation of state may be written

$$\rho = P^a \exp[-\alpha_m \theta_m (T - 1)]. \quad (47)$$

Again assuming that $\gamma = $ constant $= \gamma_m$, equations (28) and (33) yield equations (37) and (38) for ρ_0 and P_0, while T_0 now follows from equation (47) as

$$T_0 = 1 + (\alpha_m \theta_m)^{-1} a(\gamma_m - 1)(\gamma_m - a)^{-1}$$
$$\times \ln[1 - \varepsilon(\gamma_m - a)z/\gamma_m]. \quad (48)$$

The corresponding result for $\kappa_0 = c_{p0}/T_0$ is

$$\kappa_0 = \exp[(\alpha_m \theta_m)(-1 + \gamma_m/a)(T_0 - 1)/\gamma_m - 1)]. \quad (49)$$

For the case of a liquid the coefficient of $T_0 - 1$ here is of order $1/(b\phi) \gg 1$, again indicating a strong dependence of κ_0 on T_0.

Similarly, if $\alpha\theta = $ constant $= \alpha_m \theta_m \equiv b$, $K = $ constant $= K_m$, the equation of state is

$$\rho = T^{-b} \exp[K_m p_m (P - 1)]. \quad (50)$$

With $\gamma = $ constant $= \gamma_m$ the results for ρ_0 and P_0 again are given by (43) and (44), respectively, while

$$T_0 = (1 + \varepsilon K_m p_m z/\gamma_m)^{-(\gamma_m - 1)/\alpha_m \theta_m} \quad (51)$$

$$\kappa_0 = c_{p0} = T_0^{-1 - \alpha_m \theta_m/(\gamma_m - 1)}. \quad (52)$$

For the case of a liquid this again indicates a strong dependence of κ_0 on T_0, with κ_0 decreasing as T_0 is increasing.

5.2. *The perturbed state*

The perturbation equations applying to all three cases considered in the previous section are given by

$$\nabla \cdot (\rho_0 \mathbf{V}_1) = 0 \quad (53)$$

$$0 = -\frac{1}{\varepsilon\alpha_m\theta_m}\nabla P_1 - \mathbf{k}\frac{\rho_1}{\alpha_m\theta_m} - \frac{1}{Pr_m^{2/3}Gr_m^{1/2}}[\nabla\cdot\tau_0]$$

$$\tag{54}$$

$$\rho_0 c_{p0}\left[\frac{\partial T_1}{\partial t} + (\mathbf{V}_1\cdot\nabla)T_1\right] + \frac{\varepsilon\phi V_{1z}}{c_{p0}}$$

$$\times[-\rho_1 c_{p0}(\alpha\theta)_0 - \rho_0 c_{p1}(\alpha\theta)_0 + \rho_0 c_{p0}(\alpha\theta)_1]$$

$$= \frac{1}{Pr_m^{2/3}Gr_m^{1/2}}\left[\nabla\cdot(\kappa_0\nabla T_1) + \frac{\partial}{\partial z}\left(\kappa_1\frac{dT_0}{dz}\right)\right]$$

$$+ (\alpha\theta)_0\phi\left[\frac{\partial P_1}{\partial t} + (\mathbf{V}_1\cdot\nabla)P_1\right] + Q$$

$$- \varepsilon\phi\alpha_m\theta_m\frac{1}{Pr_m^{2/3}Gr_m^{1/2}}(\tau_0:\nabla\mathbf{V}_1)$$

$$\tag{55}$$

$$\rho_1 = \rho_0(K_0 p_m P_1 - \alpha_0\theta_m T_1).$$

$$\tag{56}$$

Here, $\kappa_1 = T_1 \, d\kappa(T_0)/dT_0$, $c_{p1} = T_1 \, dc_p(T_0)/dT_0$ and $(\alpha\theta)_1 = T_1 d[\alpha\theta(T_0)]/dT_0$.

An alternative set of equations may be obtained by setting

$$T_1 = T'_1 Bz,$$

$$\tag{57}$$

where B is a constant temperature gradient of the first order. This results in replacement of T_1 by T'_1 and in extra terms with B in equations (55) and (56). For a moderately deep liquid [$\varepsilon = O(1)$, $K_m p_m \ll 1$, $\phi \ll 1$, $dT_0/dz \ll 1$, $T_0 = \rho_0 = 1$], the only extra term in equation (55) is BV_{1z} on the LHS. Similarly, for an ideal gas with $c_{p0} = \kappa_0 = 1$ the only extra term in equation (55) is $\rho_0 BV_{1z}$ on the LHS. This procedure may be used to introduce an unperturbed state that is modified to first order. The corresponding replacements $\rho_1 = \rho'_1 + \rho_{1u}$ and $P_1 = P'_1 + P_{1u}$ are subject to the requirement that the perturbation quantities T'_1, ρ'_1, P'_1 and $\mathbf{V}_1$ must be zero for the unperturbed state. Substituting this requirement into the equation of state (56) and the momentum equation (54) yields the following equations for the first order unperturbed quantities:

$$\rho_{1u} = \rho_0(K_0 p_m P_{1u} - \alpha_0\theta_m Bz)$$

$$dP_{1u}/dz = -\varepsilon\rho_{1u}.$$

These equations can be solved under the boundary conditions $\rho_{1u} = P_{1u} = 0$ at $z = 0$. For a moderately deep liquid this leads to

$$\rho_1 = \rho'_1 - \alpha_m\theta_m Bz$$

$$\tag{58}$$

$$P_1 = P'_1 + \varepsilon\alpha_m\theta_m Bz^2/2,$$

$$\tag{59}$$

while for an ideal gas

$$\rho_1 = \rho'_1 + B[A(-1 + T_0^{-1} + \ln T_0) - z/T_0]\rho_0$$

$$\tag{60}$$

$$P_1 = P'_1 + BA(-1 + T_0^{-1} + \ln T_0)P_0,$$

$$\tag{61}$$

where $A \equiv \varepsilon^{-1}[\gamma_m/(\gamma_m - 1)]^2$, while ρ_0, P_0 and T_0 are given by equations (37)–(39) with $a = b = 1$. Use of these results leads to replacement of ρ_1 by ρ'_1 and of P_1 by P'_1 in equations (54)–(56).

6. STRONGLY HEATED, DEEP LIQUID WITH $Pr_m \gg 1$ AND $\alpha_m\theta_m \ll 1$

Setting $\beta = r_m v_m/L$ as in Section 3, it again follows that $\delta = \alpha_m\theta_m r_m$. Expanding equations (1)–(3) and the thermal equation of state in powers of $\alpha_m\theta_m$ yields for the unperturbed state

$$dP_0/dz = -\varepsilon\rho_0$$

$$\tag{62}$$

$$d \ln \rho_0 = K_0 p_m \, dP_0.$$

$$\tag{63}$$

These equations are essentially the same as equations (13) and (14). For $K = \text{constant} = K_m$ their solution is given by equation (14a), for $KPp_m = \text{constant} = K_m p_m$ by equation (14b). It is again assumed that P_0 and ρ_0 are independent of time. Assuming furthermore that $Pr_m r_m$ (the 'absolute' Rayleigh number) and q_m are both of order 1, the perturbed state is governed by the following set of equations:

$$\nabla\cdot(\rho_0\mathbf{V}_1) = 0$$

$$\tag{64}$$

$$0 = -\frac{1}{\varepsilon}\nabla P_1 - \mathbf{k}\rho_1 - [\nabla\cdot\tau_0]$$

$$\tag{65}$$

$$\rho_0 c_{p0}\left[\frac{\partial T_0}{\partial t} + (\mathbf{V}_1\cdot\nabla)T_0\right]$$

$$= \frac{1}{Pr_m r_m}\nabla\cdot(\kappa_0\nabla T_0) + \frac{q_m}{Pr_m r_m}Q.$$

$$\tag{66}$$

The relation between ρ_1, P_1 and T_0 is obtained as described in Section 3 [see equation (18)]. Equations closely related to those given in the present section have previously been presented by Jarvis and McKenzie [3] (see Appendix).

7. CONCLUDING REMARKS

The six sets of equations derived in the preceding pages apply to the flow of thermally driven, highly viscous fluids under various conditions. These conditions are characterized by the orders of magnitude of the parameters Pr_m, ε, $\alpha_m\theta_m$, r_m, and q_m. The inertia terms are absent from the momentum equations of all six sets. Furthermore, the thermal inertia terms are absent from the energy equations of the three sets derived in Section 3 for strongly heated fluids; these three sets are closely related. The three sets derived in Sections 4, 5 and 6 have different energy equations, while their continuity and momentum equations are essentially identical. An arbitrary heat flux of zeroth order may be added to the unperturbed state found in Section 4. The unperturbed state found in Section 5 is completely determined if the ratio of specific heats and the thermal equation of state are specified. This state is the same as that of the weakly heated, deep, slightly viscous fluid considered in ref. [1, Section 4]. It is possible in this case to add a constant temperature gradient of the first order; this results in extra terms in the energy equation.

The energy equations (8), (12), (17) and (23) of Sections 3 and 4 do not contain any derivatives with respect to time. These equations determine the

corresponding temperature fields, independent of the momentum and continuity equations. The results of Sections 3 and 4 apply to both liquids and gases. However, application of the results of Section 3 to a gas involves a very strict limitation on the depth of the gas, in view of the condition $r_m \ll 1$.

The relation $\rho_1 = -\alpha_m \theta_m T_1$, which leads to the usual form of the Boussinesq approximation, is applicable to moderately deep liquids [$K_0 p_m \ll 1$ in equations (18), (24), (56) or (67)]. For other cases, the relation between ρ_1 and T_1 involves P_1. Determination of ρ_1 then requires extraction of P_1 from the momentum equation [see equation (16) of Section 3.2, equation (22) of section 4, equation (56) of Section 5, and equation (67) of Section 6]. Similarly, determination of ρ_0 in equation (7) of Section 3 requires extraction of P_0 from the same equation. The determination of $P_{00}(t)$ appearing in Section 3.1 is discussed in ref. [1].

All sets of equations derived represent limiting cases, strictly valid only in the limit that the expansion parameter equals zero. Application to actual flows may be expected to introduce errors having the same order of magnitude as that of the expansion parameter. All sets are based on a systematic expansion procedure, involving expansion of all quantities of interest in only one parameter at a time. A second expansion is used only in Sections 3.1 and 3.2. All terms appearing in the equations are of order 1, which facilitates numerical solution.

The sets of equations of Sections 5 and 6 apply to fluids with Prandtl number much larger than one, and are applicable to geophysical flows (see Appendix and references cited therein).

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APPENDIX

McKenzie *et al.* [2] considered a fluid with infinite Prandtl number and constant specific heat, internal heating rate, coefficient of expansion, thermal conditivity and viscosity. Their equation of state (4) is based on the assumption of negligible compressibility ($K_0 p_m \ll 1$). Their main perturbation equations (10)–(12) follow term-by-term from the present equations (64)–(66) using the assumptions listed.

The equations of Jarvis and McKenzie [3] correspond to those of the present Section 6, except that they retained terms of order of the 'dissipation parameter' $Di \equiv \alpha_m \theta_m \varepsilon \phi$ in the energy equation. They considered a two-dimensional geometry. The equation of state incorporated in their equations is based on setting the Gruneisen parameter $\alpha/(\rho c_v K)$ equal to a constant. Together with the assumptions $\alpha = $ constant and $c_v = $ constant, this leads to $\rho K = $ constant $= K_m$, and hence to

$$\rho_0 = K_m p_m P_0 = \exp(-\varepsilon K_m p_m z) \qquad (A1)$$

$$\rho_1 = K_m p_m P_1 - \int_1^{T_0} \rho_0 \frac{\alpha}{\alpha_m} \, dT_0'. \qquad (A2)$$

The present equation (A1) is in agreement with the expression for the reference state used by Jarvis and McKenzie, except for a factor γ in the argument of the exponential. Because $\gamma = 1 + O(\alpha_m \theta_m)^2$ [see the thermodynamic identity in the discussion following equation (40)], this difference is negligible. The difference arises because Jarvis and McKenzie took the reference state to be adiabatic, so that $ds_0/dz = 0$. It follows from the present results that the entropy of the reference state is given by $s_0 = \int c_{p0} \, dT_0$. Taking account of the points noted, the main equations (A1)–(A4) of ref. [3] can be derived from the equations given in the present Section 6.

Turcotte *et al.* [4] chose the characteristic velocity β to be $v_m/(Di \, Pr_m L)$. They used $\delta \varepsilon \phi Pr_m$ as the expansion parameter for ρ and T, and $(\delta \varepsilon \phi Pr_m)(\varepsilon \alpha_m \theta_m)$ as the expansion parameter for P. Inclusion of the factor $\varepsilon \alpha_m \theta_m$ in the latter parameter in effect rescales the pressure P_1. Their main equations (28) and (29) follow directly from the present equations (54) and (55) for the situation considered in ref. [4]. The treatment of ref. [4] leading up to equations (28) and (29) is partly based on $Pr_m = O(1)$; this part is comparable to ref. [1, Section 4]. Equation (25) of ref. [4] contains two misprints, and misses some of the terms of the corresponding equation (26) of ref. [1].

The basic equations (1)–(4) of Torrance and Turcotte [7] can be derived from equations (54)–(58) of ref. [1] by neglecting terms of order $K_m p_m$, and setting $c_{p0} = \kappa_0 = 1$ while allowing μ_0 to be a function of T_0. The latter equations apply to a strongly heated, deep liquid with a small coefficient of thermal expansion.

The procedure followed by Turcotte *et al.* [8] is similar to that of ref. [1] for a weakly heated, deep fluid. Essentially, it consists of setting $\beta = v_m Pr_m/L$, and expanding in powers of δPr_m. This yields equations that are equivalent to equations (26)–(28) of ref. [1].

Equations (4.17)–(4.19) of Oxburgh and Turcotte [10] are closely related to equations (26)–(28) of ref. [1]. The main difference between these two sets of equations is that the term $(\partial/\partial x_i)(k\ \partial Di T_a/\partial x_i)$ in equation (4.19) is of zeroth order in the present context, and therefore does not appear in equation (28). Instead, it leads to the condition of zero vertical heat flux—equation (23a) of ref. [1] and equation (30) of the present paper. As a result, the reference adiabatic state given by equations (4.5)–(4.7) of ref. [10] is different from the unperturbed state described in Section 5.1 of the present paper. Apart from these differences, equations (4.17)–(4.19) of ref. [10] can be derived from equations (26)–(28) of ref. [1].

Peltier [11] considered a deep liquid in which the unperturbed state is maintained by a heat source distribution. His treatment basically corresponds to that of Section 4 of ref. [1]. However, he did not make use of the two forms of the conditions $ds_0/dz = 0$ [equations (32a) and (32b) of the

present paper]. These conditions follow from the requirement that the unperturbed state must be independent of the perturbation velocity V_{1z}, and retain their validity in the presence of the zeroth order heat source distribution. Equation (14) of ref. [11] contains terms with dT_0/dz and dP_0/dz. In the context of ref. [1], these terms appear in the zeroth order equation, and lead to the condition $ds_0/dz = 0$. Apart from these terms, Peltier's equation (14) can be derived from the energy equation (28) of ref. [1].

Velarde and Cordon [12] reconsidered the case studied by Turcotte *et al.* [8]. They linearized the equation of state with respect to a reference state with constant density and temperature—equation (2.5) of [12]. This is the reference state for a moderately deep liquid (see [1, Section 4]). The resulting perturbed state is governed by equations (33)–(35) of ref. [1]. The adiabatic hydrostatic reference field considered in Section 3 of ref [12, equations (3.4–3.6)] does not satisfy the conditions for the unperturbed state given in ref. [1] and in the present Section 5. The sets of equations (3.10)–(3.12) and (3.15)–(3.17) given in ref. [12] for the perturbed state incorporate the unperturbed state (3.4)–(3.6), and differ from the corresponding results (26)–(28) of ref. [1].

MOUVEMENT D'ORIGINE THERMIQUE DES FLUIDES FORTEMENT VISQUEUX

Résumé—En utilisant une méthode unifiée décrite précédemment, plusieurs systèmes d'équations sont obtenus pour le mouvement d'origine thermique de fluides fortement visqueux. Les cas couverts sont : (1) un fluide fortement chauffé; (2) un fluide peu profond, très chauffé; (3) un liquide profond, très chauffé, avec un faible coefficient de dilatation volumique; (4) un fluide profond et faiblement chauffé; (5) un fluide profond faiblement chauffé, à grand nombre de Prandtl; (6) un liquide fortement chauffé et profond avec un grand nombre de Prandtl et un faible coefficient de dilatation. Ces cas se distinguent par les ordres de grandeur des cinq paramètres suivants: le nombre de Prandtl, la profondeur adimensionnelle du fluide, le coefficient adimensionnel de dilatation, le nombre de Grashof "absolu", et le paramètre d'addition de chaleur. Dans les six systèmes d'équations obtenus, les termes d'inertie sont absents des équations de quantité de mouvement. Les termes d'inertie thermique sont absents des équations d'énergie pour les cas 1, 2 et 3. Les situations non perturbées dans les cas 3 à 6 sont supposés permanentes et de repos. Les solutions non perturbées correspondantes sont présentées en détail.

THERMISCH ANGETRIEBENE STRÖMUNGEN IN FLUIDEN MIT GROSSER ZÄHIGKEIT

Zusammenfassung—Unter Verwendung einer bereits früher beschriebenen Methode wurden verschiedene Gleichungssysteme hergeleitet, welche die thermisch angetriebene Strömung hochviskoser Fluide beschreiben. Dabei werden folgende Fälle behandelt: (1) stark beheiztes Fluid, (2) stark beheiztes Fluid geringer Tiefe, (3) stark beheiztes Fluid großer Tiefe mit geringem Volumenausdehnungskoeffizienten, (4) schwach beheiztes Fluid großer Tiefe, (5) schwach beheiztes Fluid großer Tiefe mit großer Prandtl-Zahl und (6) stark beheiztes Fluid großer Tiefe mit großer Prandtl-Zahl und kleinem Volumenausdehnungskoeffizienten. Diese Fälle unterscheiden sich um Größenordnungen bei der Prandtl-Zahl, der dimensionslosen Tiefe des Fluids, dem dimensionslosen Volumenausdehnungskoeffizienten, der "absoluten" Grashof-Zahl und dem Heizparameter. In allen sechs Gleichungssystemen entfällt die Trägheitskraft in den Bewegungsgleichungen. Die Speicherterme entfallen in den Energiegleichungen bei den Fällen (1), (2) und (3). Der ungestörte Zustand in den Fällen (3) bis (6) wurde als stationär und ruhend angenommen. Entsprechende Lösungen werden detailliert dargestellt.

ДВИЖЕНИЕ СИЛЬНОВЯЗКИХ ЖИДКОСТЕЙ, ВЫЗВАННОЕ НАГРЕВОМ

Аннотация—Несколько систем уравнений движения сильновязких жидкостей выведены с использованием формального унифицированного метода, описанного в предыдущей работе. Рассматриваются случаи сильно нагретой жидкости (I), слоя малой толщины сильно нагретой жидкости (II), слоев большой толщины сильно нагретой жидкости с малым коэффициентом объемного расширения (III), слабо нагретой жидкости (IV), слабо нагретой жидкости с высоким значением числа Прандтля (V) и сильно нагретой жидкости с большим числом Прандтля и малым коэффициентом объемного расширения (VI). Указанные случаи различаются между собой по порядку величин числа Прандтля, размерной толщины слоя, размерного коэффициента объемного расширения, "абсолютного" числа Грасгофа и параметра температурного напора. Все уравнения импульса не содержат инерционные члены, а энергии для случаев (I), (II) и (III)—тепловые инерционные члены. Невозмущенные состояния в случаях (III)–(VI) полагаются установившимися и устойчивыми. Приведены соответствующие невозмущенные решения.